An alternative expression for the Black-Scholes formula in terms of Brownian first and last passage times

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February 11, 2008

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Abstract: The celebrated Black-Scholes formula which gives the price of a European option, may be expressed as the cumulative function of a last passage time of Brownian motion. A related result involving first passage times is also obtained.

Keywords: Black-Scholes formula, last passage times, time-inversion, Cameron-Martin relationship, reflection principle.

1 Introduction and main results

► **a)** Let $(B_t, t \geq 0)$ denote a one-dimensional Brownian motion starting from 0, and let

$$\mathcal{E}_t = \exp\left(B_t - \frac{t}{2}\right), \ (t \geq 0)$$

A reduced form of the celebrated Black-Scholes formula is the following.

$$(\mathbf{1_K}) \qquad E[(\mathcal{E}_t - K)^+] = \mathcal{N}\left(-\frac{\ell n\, K}{\sqrt{t}} + \frac{\sqrt{t}}{2}\right) - K\mathcal{N}\left(-\frac{\ell n\, K}{\sqrt{t}} - \frac{\sqrt{t}}{2}\right)$$

where $K \geq 0$, and as usual:

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} dy \, \exp\left(-\frac{y^2}{2}\right)$$

In fact, formula $(\mathbf{1_K})$ may be split into two parts:

$$(\mathbf{1_K^+}) \qquad E[\mathcal{E}_t \, 1_{\mathcal{E}_t > K}] = \mathcal{N}\left(-\frac{\ell n\, K}{\sqrt{t}} + \frac{\sqrt{t}}{2}\right)$$

$$(\mathbf{1_K^-}) \qquad K\, P(\mathcal{E}_t > K) = K\, \mathcal{N}\left(-\frac{\ell n\, K}{\sqrt{t}} - \frac{\sqrt{t}}{2}\right)$$

As noted in Section 5 of [1], formula $(\mathbf{1_K^\pm})$ is obtained in an elementary manner, after performing the change of probability:

$$P'_{|\mathcal{F}_t} = \mathcal{E}_t {\scriptstyle\bullet} P_{|\mathcal{F}_t}$$

which transforms (B_t) in $(B_t + t)$ hence $\left(B_t - \frac{t}{2}\right)$ in $\left(B_t + \frac{t}{2}\right)$.

► **b)** In this Note, we give and discuss a different representation of $(\mathbf{1_K^\pm})$.

Theorem 1. *For any $K \geq 0$, there are the representations:*

$$(\mathbf{2_K^-}) \qquad E[\mathcal{E}_t \, 1_{(\mathcal{E}_t > K)}] - K\, P(\mathcal{E}_t > K) = P\left(G^{(1/2)}_{(\ell n\, K)} \leq t\right) \quad (K \geq 0)$$

$$(\mathbf{2^+_{K \geq 1}}) \qquad E[\mathcal{E}_t \, 1_{\mathcal{E}_t > K}] + K\, P(\mathcal{E}_t > K) = P\left(T^{(1/2)}_{(\ell n\, K)} \leq t\right) \quad (K \geq 1)$$

$$(\mathbf{2^+_{K \leq 1}}) \qquad E[\mathcal{E}_t \, 1_{\mathcal{E}_t < K}] + K\, P(\mathcal{E}_t < K) = P\left(T^{(1/2)}_{(\ell n\, K)} \leq t\right) \quad (0 \leq K \leq 1)$$

or, equivalently:

$$(\mathbf{2^{++}_{K\le 1}})\quad E[\mathcal{E}_t\, 1_{\mathcal{E}_t>K}] + K\, P(\mathcal{E}_t > K) = K + P\left(T^{(1/2)}_{(\ell n\, K)} > t\right) \quad (0 \le K \le 1)$$

where, for $\nu \in \mathbb{R}$, and $B^{(\nu)}_t \equiv B_t + \nu t$, we write:

$$T^{(\nu)}_a = \inf\{t : B^{(\nu)}_t = a\}\,;\; G^{(\nu)}_a = \sup\{t : B^{(\nu)}_t = a\}$$

Comment and Complements about Theorem 1:

(i) Our motivation to prove formulae such as $(\mathbf{2^{-}_{K}})$ was our desire to obtain an expression on the RHS showing in a clear manner that the LHS $(= E((\mathcal{E}_t - K)_+))$ is an increasing function of t.
This is not clear from $(\mathbf{1_K})$, although this property of increase is a consequence of the submartingale property of $(\mathcal{E}_t - K)^+$; see Section 4 for a more extended discussion.

(ii) Obviously, an equivalent presentation of the "system" $(\mathbf{2^{\pm}_{K}})$ is, for $K \ge 1$:

$$(\mathbf{3^{+}_{K\ge 1}})\qquad E[\mathcal{E}_t\, 1_{(\mathcal{E}_t>K)}] = \frac{1}{2}\left\{P\left(T^{(1/2)}_{(\ell n\, K)} \le t\right) + P\left(G^{(1/2)}_{(\ell n\, K)} \le t\right)\right\}$$

$$(\mathbf{3^{-}_{K\ge 1}})\qquad K\, P(\mathcal{E}_t > K) = \frac{1}{2}\left\{P\left(T^{(1/2)}_{(\ell n\, K)} \le t\right) - P\left(G^{(1/2)}_{(\ell n\, K)} \le t\right)\right\}$$

$$= \frac{1}{2}\, P\left(T^{(1/2)}_{(\ell n\, K)} \le t \le G^{(1/2)}_{(\ell n\, K)}\right)$$

and, for $0 \le K \le 1$:

$$(\mathbf{3^{+}_{K\le 1}})\qquad E[\mathcal{E}_t\, 1_{(\mathcal{E}_t>K)}] = \frac{1}{2}\left\{1 + K - P\left(T^{(1/2)}_{(\ell n\, K)} \le t \le G^{(1/2)}_{(\ell n\, K)}\right)\right\}$$

$$(\mathbf{3^{-}_{K\le 1}})\quad K\, P(\mathcal{E}_t > K) = \frac{1}{2}\left\{1 + K - \left[P\left(T^{(1/2)}_{(\ell n\, K)} \le t\right) + P\left(G^{(1/2)}_{(\ell n\, K)} \le t\right)\right]\right\}$$

(iii) In order to give formulae $(\mathbf{2^{\pm}_{K}})$ an "explicit" character, we now recall the distributions of

$$T^{(\nu)}_a = \inf\{t : B^{(\nu)}_t = a\} \text{ and } G^{(\nu)}_a = \sup\{t : B^{(\nu)}_t = a\}\ ,$$

for $\nu > 0$, and $a > 0$ (these formulae will then be used with $\nu = \frac{1}{2}$ and $a = \ell n\, K$): denoting by $p^{(\nu)}_t(a)$ the density of $B^{(\nu)}_t$, we have:

$$(4)\qquad P(T^{(\nu)}_a \in dt) = \left(\frac{a}{t}\right) p^{(\nu)}_t(a) dt \equiv \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}\,(a - \nu t)^2\right) dt$$

whereas

$$(\mathbf{5}) \qquad P(G_a^{(\nu)} \in dt) = \nu p_t^{(\nu)}(a)dt \equiv \frac{\nu}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(a-\nu t)^2\right) dt$$

Formula (**4**) may be obtained from the combination of (**4**) for $\nu = 0$, which is very well known, followed by an application of the Cameron-Martin relationship between the laws of $B^{(\nu)}$ and B. Formula (**5**) is a particular case of a more general formula for last passage times of transient diffusions, obtained in Pitman-Yor [3].

(iv) Although this is not strictly necessary at this point (but will be useful in our proof of Theorem 1), we also present the distributions of $T_a^{(-\nu)}$ and $G_a^{(-\nu)}$, for $a \geq 0$.
In fact, they may be obtained easily from those of $T_a^{(\nu)}$ and $G_a^{(\nu)}$ thanks to the Cameron-Martin absolute continuity relationships:

$$(\mathbf{6}) \qquad \begin{aligned} W^{(-\nu)}_{|\mathcal{F}_{T_a}\cap(T_a<\infty)} &= \exp(-2\nu a) \bullet W^{(\nu)}_{|\mathcal{F}_{T_a}} \\ W^{(-\nu)}_{|\mathcal{F}_{G_a}\cap(G_a>0)} &= \exp(-2\nu a) \bullet W^{(\nu)}_{|\mathcal{F}_{G_a}} \end{aligned}$$

where, here, $W^{(\mu)}$ denotes the law of $(B_t + \mu t, t \geq 0)$ on canonical space, and T_a, resp. G_a, is the first, resp. last, hitting time of a by the coordinate process.
Thus, we deduce from formulae (**6**), (**4**) and (**5**) that:

$$P(G_a^{(-\nu)} > 0) = P(T_a^{(-\nu)} < \infty) = \exp(-2\nu a)$$

whereas:

$$(\mathbf{7}) \qquad \begin{aligned} P(T_a^{(-\nu)} \in dt) &= \left(\tfrac{a}{t}\right) p_t^{(-\nu)}(a)dt = \tfrac{a}{\sqrt{2\pi t^3}} \exp\left(-\tfrac{1}{2t}(a+\nu t)^2\right) dt \\ P(G_a^{(-\nu)} \in dt) &= \nu p_t^{(-\nu)}(a)dt = \tfrac{\nu}{\sqrt{2\pi t}} \exp\left(-\tfrac{1}{2t}(a+\nu t)^2\right) dt \end{aligned}$$

▶ **c)** Organisation of the remainder of the paper:

- In Section 2, we prove Theorem 1, independently from formulae $(\mathbf{1}^{\pm}_{\mathbf{K}})$
- In Section 3, we give an elementary proof of the agreement between formulae $(\mathbf{1}^{\pm}_{\mathbf{K}})$ and $(\mathbf{3}^{\pm}_{\mathbf{K}})$
- Section 4 concludes, by setting the matter in a broader context.

2 Proof of Theorem 1

Clearly, in order to prove Theorem 1, it now suffices to prove $(\mathbf{2_K^-})$, $(\mathbf{3^-_{K\geq 1}})$ and $(\mathbf{2^+_{K\leq 1}})$.

► **a)** Proof of $(\mathbf{2_K^-})$ (for any $K \geq 0$)
We shall show that

$$E[(\mathcal{E}_t - K)^+] \underset{(i)}{=} K\,P\left(0 < G^{(-1/2)}_{(\ell n\, K)} \leq t\right) \underset{(ii)}{=} P\left(G^{(+1/2)}_{(\ell n\, K)} \leq t\right)$$

The proof of (ii) follows from the relationship between the laws of $G_a^{(-\nu)}$ and $G_a^{(\nu)}$ as discussed in Section 1; namely:

$$P(G_a^{(-\nu)} > 0) = \exp(-2\nu a) \text{ and } P(G_a^{(-\nu)} \in dt | G_a^{(-\nu)} > 0) = P(G_a^{(\nu)} \in dt)$$

For the proof of (i), we rely upon the following formula

$$(\mathbf{8}) \qquad P(G_a^{(\mu)} \geq t | \mathcal{F}_t) = \left(\frac{\exp(2\mu a)}{\exp(2\mu B_t^{(\mu)})}\right) \wedge 1$$

which is valid for all $\mu \in \mathbb{R}$; this is a particular case of the results for last passage times of a transient real-valued diffusion, as discussed in Pitman-Yor [3].
In particular, for $\mu = -\nu$, $\nu > 0$, we get:

$$\exp(2\nu a) P(0 < G_a^{(-\nu)} \leq t | \mathcal{F}_t) = (\exp(2\nu B_t^{(-\nu)}) - \exp(2\nu a))^+$$

This obviously proves (i), by taking $a = (\ell n\, K)$, $\nu = 1/2$.

► **b)** Proof of $(\mathbf{3^-_{K\geq 1}})$ (for $K \geq 1$)
Using again formula $(\mathbf{8})$, we see that $(\mathbf{3^-_{K\geq 1}})$ is equivalent to:

$$(\mathbf{9_K}) \quad K\,P\left(B_t - \frac{t}{2} > (\ell n\, K)\right) = \frac{1}{2} E\left(1_{(T^{(1/2)}_{(\ell n\, K)} \leq t)} \bullet \left(\frac{K}{\exp\left(B_t + \frac{1}{2}\right)} \wedge 1\right)\right)$$

We now use the Cameron-Martin relationship on both sides to reduce the statement of $(\mathbf{9_K})$ to a statement about standard Brownian motion (B_t), for which we denote: $M_t = \sup_{s\leq t} B_s$. Thus, we find that $(\mathbf{9_K})$ is equivalent to:

$$(\mathbf{10_K}) \qquad K\,E\left(1_{(B_t > (\ell n\, K))} \bullet \exp\left(-\frac{B_t}{2}\right)\right)$$

$$= \frac{1}{2}\, E\left(1_{(M_t>(\ell n\, K))} \bullet \left(\frac{K}{\exp(B_t)} \wedge 1\right) \bullet \exp\left(\frac{B_t}{2}\right)\right)$$

We now decompose the RHS of $(\mathbf{10_K})$ in a sum of two quantities:

$$\frac{1}{2} \quad \left\{ \begin{array}{l} E\left[1_{(M_t>(\ell n\, K))} \bullet 1_{(B_t>\ell n\, K)} \bullet K \exp\left(-\frac{B_t}{2}\right)\right] \\ \\ +E\left[1_{(M_t>\ell n\, K)} \bullet 1_{(B_t<\ell n\, K)} \bullet \exp\left(+\frac{B_t}{2}\right)\right] \end{array} \right\}$$

Thus, $(\mathbf{10_K})$ now gets simplified to the equivalent form:

$$(\mathbf{11_K}) \qquad \frac{K}{2}\, E\left(1_{(B_t>(\ell n\, K))} \bullet \exp\left(-\frac{B_t}{2}\right)\right)$$

$$= \frac{1}{2}\, E\left(1_{(M_t>(\ell n\, K))} \bullet 1_{(B_t<\ell n\, K)} \bullet \exp\left(\frac{B_t}{2}\right)\right)$$

which, taking $x = (\ell n\, K)$, may be written as:

$$(\mathbf{12}_x) \qquad E\left(1_{(B_t>x)} \bullet \exp\left(x - \frac{B_t}{2}\right)\right) = E\left(1_{(M_t>x>B_t)} \bullet \exp\left(\frac{B_t}{2}\right)\right)$$

We now show $(\mathbf{12}_x)$, from the right to the left, as a consequence of the reflection principle:
conditionally on $\mathcal{F}_{T_x}$, and $T_x < t$, we have:

$$B_t - x = \widehat{B}_{(t-T_x)}\,, \text{ with } \widehat{B} \text{ independent from } \mathcal{F}_{T_x}\,;$$

hence, under this condition, the reflection principle boils down to:

$$(\maltese) \qquad B_t - x \overset{\text{(law)}}{=} -(B_t - x)$$

Thus, the RHS of $(\mathbf{12}_x)$ is:

$$\begin{aligned} & E\left(1_{(T_x<t)} \bullet 1_{(B_t-x<0)} \bullet \exp\left(\frac{1}{2}\left\{x + (B_t - x)\right\}\right)\right) \\ \underset{(\text{from } (\maltese))}{=} \; & E\left(1_{(T_x<t)} \bullet 1_{(B_t-x<0)} \bullet \exp\left(\frac{1}{2}\left\{x - (B_t - x)\right\}\right)\right) \\ = \; & E\left(1_{(B_t>x)} \bullet \exp\left(x - \frac{B_t}{2}\right)\right)\,, \text{ which is the LHS of } (\mathbf{12}_x) \end{aligned}$$

This proves $(\mathbf{3}_{K\geq 1})$, and, with $(\mathbf{2}_K^-)$, $(\mathbf{2}_{K\geq 1}^+)$.

► **c)** <u>We now prove that</u> $(\mathbf{2}^{+}_{K\geq 1})$ <u>implies</u> $(\mathbf{2}^{+}_{K\leq 1})$ (for $0 \leq K \leq 1$):
We introduce the probability P' such that:

$$P'_{\mathcal{F}_t} = \mathcal{E}_t \bullet P_{\mathcal{F}_t}$$

We note that, under P', $\dfrac{1}{\mathcal{E}_t} := \widehat{\mathcal{E}}_t = \exp\left(\widehat{B}_t - \dfrac{t}{2}\right)$, for a new Brownian motion $(\widehat{B}_t, t \geq 0)$. Thus, the LHS of $(\mathbf{2}^{+}_{K\leq 1})$ writes:

$$\begin{aligned}
& P'(\mathcal{E}_t < K) + K\, P'(\widehat{\mathcal{E}}_t - 1_{(\mathcal{E}_t<K)}) \\
= \;& P'\left(\widehat{\mathcal{E}}_t > \frac{1}{K}\right) + K\, E'\left(1_{(\widehat{\mathcal{E}}_t > \frac{1}{K})}\right) \\
= \;& K\, E'\left(1_{(\widehat{\mathcal{E}}_t > \frac{1}{K})}\right) + \frac{1}{K} P'\left(\widehat{\mathcal{E}}_t > \frac{1}{K}\right) \\
= \;& K\, P\left(T^{(1/2)}_{(\ell n\, K)} \leq t\right) \quad (\text{from}(\mathbf{2}^{+}_{\frac{1}{K}})) \\
= \;& K\, P\left(T^{(-1/2)}_{(\ell n\, K)} \leq t\right) \quad (\text{by symmetry}) \\
= \;& P\left(T^{(1/2)}_{(\ell n\, K)} \leq t\right) \quad (\text{from}(6))
\end{aligned}$$

► **d)** <u>Finally, we observe that</u> $(\mathbf{2}^{+}_{K\leq 1})$ <u>is equivalent to</u> $(\mathbf{2}^{++}_{K\leq 1})$, <u>since:</u>

$$\begin{aligned}
& E(\mathcal{E}_t 1_{(\mathcal{E}_t<K)}) + K\, P(\mathcal{E}_t < K) \\
= \;& 1 - E(\mathcal{E}_t 1_{(\mathcal{E}_t>K)}) + K(1 - P(\mathcal{E}_t > K)) \; (\text{since } E(\mathcal{E}_t) = 1) \\
= \;& 1 + K - \{E(\mathcal{E}_t 1_{(\mathcal{E}_t>K)}) + K\, P(\mathcal{E}_t > K)\}
\end{aligned}$$

3 On the agreement between the classical Black-Scholes formula $(\mathbf{1}^{\pm}_{\mathbf{K}})$ and our main result

► **a)** We now check in an elementary manner formulae $(\mathbf{2}^{\pm}_{\mathbf{K}})$ by comparing their LHS, as given from the "traditional" Black-Scholes formulae $(\mathbf{1}^{\pm}_{\mathbf{K}})$, with their RHS, as given by $(\mathbf{4})$ and $(\mathbf{5})$.

► **b)** The case $K \geq 1$. Since both sides of $(\mathbf{2}^{-}_{\mathbf{K}})$ and $(\mathbf{2}^{+}_{K\geq 1})$ are equal to 0 for $t = 0$, we need only check that the derivatives in t are equal; thus, our task is to show:

$$(\mathbf{13}^{-}_{\mathbf{K}}) \qquad \frac{\partial}{\partial t}\left\{\mathcal{N}\left(-\frac{\ell n\, K}{\sqrt{t}} + \frac{\sqrt{t}}{2}\right) - K\mathcal{N}\left(-\frac{\ell n\, K}{\sqrt{t}} - \frac{\sqrt{t}}{2}\right)\right\}$$

$$= \left(\frac{K}{2}\right) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}\left((\ell n\, K) + \frac{t}{2}\right)^2\right)$$

$$(\mathbf{13^+_{K\geq 1}}) \qquad \frac{\partial}{\partial t}\left\{\mathcal{N}\left(-\frac{\ell n\, K}{\sqrt{t}} + \frac{\sqrt{t}}{2}\right) + K\mathcal{N}\left(-\frac{\ell n\, K}{\sqrt{t}} - \frac{\sqrt{t}}{2}\right)\right\}$$

$$= K(\ell n\, K)\frac{1}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}\left((\ell n\, K) + \frac{t}{2}\right)^2\right)$$

(We also see on these expressions the relationships between $p_t^{(-1/2)}(x)$ and $p_t^{(+1/2)}(x)$).
To prove $(\mathbf{13^-_K})$ and $(\mathbf{13^+_{K\geq 1}})$, we compute:

- $$\frac{\partial}{\partial t}\left(\mathcal{N}\left(-\frac{\ell n\, K}{\sqrt{t}} + \frac{\sqrt{t}}{2}\right)\right)$$
$$= \frac{\sqrt{K}}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{(\ell n\, K)^2}{t} + \frac{t}{4}\right)\right)\left(\frac{\partial}{\partial t}\left\{-\frac{(\ell n\, K)}{\sqrt{t}} + \frac{\sqrt{t}}{2}\right\}\right)$$
- $$K\frac{\partial}{\partial t}\left(\mathcal{N}\left(-\frac{\ell n\, K}{\sqrt{t}} - \frac{\sqrt{t}}{2}\right)\right)$$
$$= \frac{\sqrt{K}}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{(\ell n\, K)^2}{t} + \frac{t}{4}\right)\right)\left(\frac{\partial}{\partial t}\left\{-\frac{(\ell n\, K)}{\sqrt{t}} - \frac{\sqrt{t}}{2}\right\}\right)$$

and $(\mathbf{13^-_K})$ and $(\mathbf{13^+_{K\geq 1}})$ are then obtained by elementary algebraic manipulations. (In fact, it is these very manipulations which led us to believe in the truth of Theorem 1!!).

▶ **c)** The case $0 \leq K \leq 1$. Since both sides of $(\mathbf{2}^{++}_{K\leq 1})$ are equal to $1 + K$ for $t = 0$, it suffices to prove, for $K \leq 1$:

$$\frac{\partial}{\partial\, t}\left\{\mathcal{N}\left(-\frac{log\, K}{\sqrt{t}} + \frac{\sqrt{t}}{2}\right) + K\mathcal{N}\left(-\frac{log\, K}{\sqrt{t}} - \frac{\sqrt{t}}{2}\right)\right\}$$
$$= \frac{\partial}{\partial\, t} P\left(T^{(1/2)}_{(\ell n\, K)} > t\right)$$

and this latter relation follows immediately from the computations done in point **b)** above and from (**7**).
Note also that $(\mathbf{2}^{+}_{K\geq 1})$ and $(\mathbf{2}^{+}_{K\leq 1})$ coincide for $K = 1$.

4 Further remarks and conclusion

► **a)** Easy variants of formulae ($\mathbf{2_K^{\pm}}$) may be written, e.g. by using the scaling property of Brownian motion, so that $T_a^{(\nu)}$ and $G_a^{(\nu)}$ appear on the RHS of ($\mathbf{2_K^{\pm}}$); however, writing down these variants would only complicate unnecessarily these formulae.

► **b)** We recall that, as a consequence of the time inversion property of Brownian motion, there are the relations:

$$(T_a^{(\nu)}\ ,\ G_a^{(\nu)}) \overset{\text{(law)}}{=} \left(\frac{1}{G_\nu^{(a)}}\ ,\ \frac{1}{T_\nu^{(a)}} \right)$$

(see e.g., Pitman-Yor [3] for a more general discussion).
In particular, for $a = 0$, one gets:

$$G_0^{(\nu)} \overset{\text{(law)}}{=} \frac{1}{T_\nu^{(0)}} \overset{\text{(law)}}{=} \frac{B_1^2}{\nu^2}$$

In particular, formula ($\mathbf{2^-_{K=1}}$) becomes:

$$(\mathbf{14_1}) \qquad E[(\mathcal{E}_t - 1)^+] = E[(\mathcal{E}_t - 1)^-] = P(4B_1^2 \le t)$$

which allowed us to answer M. Qian's question [4]:
is there a simple formula for:

$$\int_0^\infty \theta(dt) E[(\mathcal{E}_t - 1)^{\pm}]$$

where $\theta(dt)$ is a probability on $\mathbb{R}_+$?
From ($\mathbf{14_1}$), we easily obtain:

$$(\mathbf{15}) \qquad \int_0^\infty \theta(dt) E[(\mathcal{E}_t - 1)^{\pm}] = E[\overline{\theta}(4B_1^2)]$$

where $\overline{\theta}(x) = \theta([x,\infty))$ is the tail of θ.
To particularise even more, we give the explicit form of the Laplace transform:

$$(\mathbf{16}) \qquad \int_0^\infty dt\, e^{-\lambda t}\, E[(\mathcal{E}_t - 1)^{\pm}] = \frac{1}{\lambda}\, E[\exp(-\lambda(4B_1^2))]$$

$$= \frac{1}{\lambda} \frac{1}{\sqrt{1+8\lambda}}$$

It is this question which set us on the general quest for a representation of

$$E[(\mathcal{E}_t - K)^+]$$

as a cumulative distribution function in t.

► **c)** We come back to the time inversion property of BM, in order to throw another light upon our main result $(\mathbf{2}_K^-)$, which relates the European call price with the cumulative function of last Brownian passage times. (This paragraph has been partly inspired by unpublished notes by Peter Carr [2].) Indeed, a variant of $(\mathbf{2}_K^-)$ is the following:
for every $t \geq 0$, $K \geq 0$, and $\phi : C([0,t]) \to \mathbb{R}_+$, measurable,

$$(\mathbf{17}) \qquad E[\phi(B_u, u \leq t)(K - \mathcal{E}_t)^+] = K\, E[\phi(B_u, u \leq t) 1_{(\mathcal{G}_K \leq t)}]$$

where $\mathcal{G}_K = \sup\{u : \mathcal{E}_u = K\}$.
Writing $(\mathbf{17})$ in terms of the Brownian motion $(\widehat{B}_v, v \geq 0)$ such that: $B_u = u\widehat{B}_{(1/u)}$, and setting $s = 1/t$, it is clearly seen that $(\mathbf{17})$ is equivalent to:

$$K\, P(\widehat{T}_{1/2}^{(-\ell n K)} \geq s | \widehat{B}_s) = \left(K - \exp\left(\frac{1}{s} \widehat{B}_s - \frac{1}{2s} \right) \right)^+$$

where: $\widehat{T}_a^{(\nu)} = \inf\{u : \widehat{B}_u + \nu u = a\}$.
Since hats are no longer necessary for our purpose, we drop them, and we now look for an independent proof of:

$$(\mathbf{18}) \qquad P(T_{1/2}^{(-\ell n K)} \geq s | B_s = x) = \left(1 - \frac{1}{K} \exp\left(\frac{x}{s} - \frac{1}{2s} \right) \right)^+$$

On the LHS of $(\mathbf{18})$, we may replace $(B_s = x)$ by $(B_s - s(\ell n K) = x - s(\ell n K))$. Now, as a consequence of the Cameron-Martin relationship, the conditional expectation:

$$E[F(B_u - \nu u, u \leq s) | B_s - \nu s = y]$$

does not depend on ν; hence, $(\mathbf{18})$ is equivalent to:

$$P(T_{1/2}^{(-\ell n K)} \geq s | B_s = x - s(\ell n K)) = \left(1 - \frac{1}{K} \exp\left(\frac{x - \frac{1}{2}}{s} \right) \right)^+$$

which simplifies to:

$$P(\sup_{u \leq s} B_u < \frac{1}{2} | B_s = y) = \left(1 - \exp\left(\frac{y - \frac{1}{2}}{s} \right) \right)^+$$

or, by scaling:

$$P(\sup_{u\leq 1} B_u < \frac{1}{2\sqrt{s}} | B_1 = \frac{y}{\sqrt{s}}) = \left(1 - \exp\left(\frac{1}{\sqrt{s}}\left(\frac{y}{\sqrt{s}} - \frac{1}{2\sqrt{s}}\right)\right)\right)^+$$

This is equivalent to:

$$(\mathbf{19}) \qquad P(\sup_{u\leq 1} B_u < \sigma | B_1 = y) = (1 - \exp(2\sigma(y-\sigma))^+$$

for $\sigma \geq 0$, and $y \in \mathbb{R}$.
This formula is trivial for $\sigma < y$, and, for $\sigma \geq y$, it follows from the classical formula:

$$(\mathbf{20}) \qquad P(\sup_{u\leq 1} B_u \in d\sigma, B_1 \in da) = \frac{da\, d\sigma}{\sqrt{2\pi}} 2(2\sigma - a) e^{-\frac{(2\sigma-a)^2}{2}} 1_{\{a<\sigma;\sigma\geq 0\}}$$

▶ **d)** In a future work, we plan to study more generally how quantities such as the calls and puts:

$$E[(S_t - K)^+] \text{ and } E[(S_t - K)^-]$$

associated with a general $\mathbb{R}_+$-valued continuous local martingale $(S_t, t \geq 0)$ may be written in terms of cumulative functions.

▶ **e)** In [1], the authors present eight different approaches to the Black-Scholes formula, among which the change of numéraire approach (Section 5 of [1]), and the local time approach (Section 6 of [1]). This local time approach, together with $(\mathbf{2_K^-})$ yields the relationship:

$$(21) \qquad P\left(G^{(1/2)}_{(\ell n\, K)} \leq t\right) = \left(\frac{1}{2}\right) E[\mathcal{L}_t^K]$$

where $(\mathcal{L}_t^K, t \geq 0)$ denotes the local time at level K of $(\mathcal{E}_t, t \geq 0)$. It is this kind of relationship (**21**) which is central in the obtention in [3] of a general expression for the law of a last passage time of a transient diffusion. However, to our knowledge, despite the remarkable survey [1] of methods leading to the Black-Scholes formula, no interpretation of this formula seems to have been made in terms of last passage times distributions.

References

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Acknowledgments: We are grateful to M. Qian for raising the question mentioned in Section 4, and to T. Fujita for pointing out reference [1].